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Stabilized element residual method (SERM): A posteriori error estimation for the advection–diffusion equation

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Abstract

Residual-based a posteriori error estimation techniques have been developed for linear elliptic symmetric positive-definite problems. One asymptotically-exact error estimator for the elliptic Laplacian operator relies on solving local Neumann problems in each element. This technique is extended to the unsymmetric and positive semi-definite advection–diffusion (AD) operator. Here, a novel approach, the Stabilized Element Residual Method (SERM) is presented. In this method, the unsymmetric advection terms are retained in the formulation of the local error problem through the use of stabilized methods. The selection of the optimum stabilization parameter is discussed.

Keywords: Adaptive finite element method; A posteriori error estimation; Advection–diffusion; Stabilized Element Residual Method

1. Introduction

A posteriori error estimates provide a quantitative measure of the error on a particular mesh and are used to drive the mesh refinement in an adaptive finite element method. An extensive review of the literature, containing a variety of *a posteriori* error estimators, is available in the papers by Oden et al. [12, 13] and Ainsworth and Oden [2]. The residual-based approaches have the potential of being applicable to a large class of problems.

It is convenient to divide residual-based estimators into implicit and explicit categories. The implicit estimators involve the solution of local boundary value problems whose variational form is identical to that of the global boundary value problem. Babuska and Rheinboldt [3] presented the subdomain residual method where the local problem is posed over a patch of elements while Demkowicz et al. [5] and Bank and Weiser [4] simultaneously proposed the element residual method (ERM) that poses the local problem over a single element. Explicit estimators, on the other hand, are computed directly from the solution and the given data of the problem. This scheme does not

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require solving any system of equations. These have been proposed, for example, by Johnson and Eriksson [6, 11].

Although the explicit approach is computationally efficient, it involves unknown constants that have to be determined by solving global dual problems [15]. Another difficulty is determining the relative weighting of the interior and boundary residual terms. Implicit methods circumvent these issues by solving local problems with the residuals as source terms. Also, while the explicit schemes provide an estimate of the error in a particular norm, the implicit methods have the flexibility of choosing any suitable norm. However, one has to deal with the issues of the proper choice of local subspaces for the error and flux splitting factors.

In this paper we develop implicit element residual-type estimates for application in the adaptive solution of the scalar steady advection–diffusion (AD) equation. While numerous authors have considered *a posteriori* error estimates for linear, elliptic, self-adjoint and positive-definite problems, the AD equation has received limited treatment. Strouboulis and Oden [16] used the Taylor–Galerkin type operator-splitting technique to obtain the solution and ERM to measure the error in the advective and diffusive steps separately. Explicit residual estimates were proposed in [7, 11] where the shock-capturing streamline diffusion method (SDM) is the underlying method used to obtain the numerical solution. Ainsworth and Oden [1] presented an approach that makes use of duality arguments and is based on the ERM. For unsymmetrical problems, they formulated the local problem by introducing a symmetric, positive-definite bilinear form involving arbitrary constants.

In this chapter we present the *stabilized element residual method* (SERM) where the local error problem inherits the same bilinear operator as in the global problem. This technique retains the unsymmetric (advection) terms through the use of a stabilized method for the local error problem. The method incorporates a stabilization parameter that is selected to achieve optimal accuracy. The end result is an error indicator that is valid in both diffusion-dominated and advection-dominated limits. The optimal stabilization parameter is computed for a model 1-D problem. A straightforward application of this 1-D parameter in a 2-D example provides good results.

2. Problem statement and notation

Let Ω be an open, bounded region in $\mathbb{R}^d$, where d is the number of space dimensions. The boundary of Ω is denoted by Γ and is assumed smooth. The unit outward normal vector to Γ is denoted by $\hat{\boldsymbol{n}}$. Let $\boldsymbol{a}$ denote the given flow velocity, assumed solenoidal, i.e., $\nabla \cdot \boldsymbol{a} = 0$. The following notations are useful:

$$a_n = \hat{\boldsymbol{n}} \cdot \boldsymbol{a} \tag{1}$$

$$a_n^+ = \tfrac{1}{2}(a_n + |a_n|) \tag{2}$$

$$a_n^- = \tfrac{1}{2}(a_n - |a_n|) \tag{3}$$

Let Γ^-, Γ^+ and Γ_g, Γ_h be partitions of Γ, where

$$\Gamma^- = \{x \in \Gamma | a_n(x) < 0\} \quad \text{(inflow boundary)}, \tag{4}$$

$$\Gamma^+ = \Gamma - \Gamma^- \quad \text{(outflow boundary)}. \tag{5}$$

Let $\kappa > 0$ denote the diffusivity. The scalar steady advection–diffusion equation consists of finding $u = u(\boldsymbol{x})\ \forall\ \boldsymbol{x} \in \bar{\Omega} \ni$

$$\mathscr{L}u \equiv \boldsymbol{a} \cdot \nabla u - \nabla \cdot (\kappa \nabla u) = f \quad \text{in } \Omega, \tag{6}$$

$$u = g \quad \text{on } \Gamma_g, \tag{7}$$

$$-a_n^- u + \hat{\boldsymbol{n}} \cdot \kappa \nabla u = h \quad \text{on } \Gamma_h, \tag{8}$$

where $f : \Omega \to \mathbb{R}$, $g : \Omega_g \to \mathbb{R}$, and $h : \Gamma_h \to \mathbb{R}$ are prescribed data. The advective component of the differential operator $\mathscr{L}$ is denoted by $\mathscr{L}_{\text{adv}}$,

$$\mathscr{L}_{\text{adv}}\, u \equiv \boldsymbol{a} \cdot \nabla u. \tag{9}$$

The adjoint of $\mathscr{L}$ is denoted by $\mathscr{L}^*$. Here

$$\mathscr{L}^* u \equiv -\boldsymbol{a} \cdot \nabla u - \nabla \cdot (\kappa \nabla u) \tag{10}$$

The element Peclet number, α, which will appear frequently in this chapter, is defined as

$$\alpha = \frac{|\boldsymbol{a}|h}{2\kappa}, \tag{11}$$

where h is the element mesh size defined in [10] for multi-dimensions.

3. Stabilized methods (SM) formulation

It is a well-established fact that the Galerkin method yields oscillatory solutions for $\alpha \geqslant 1$. The stabilized methods impart stability to the Galerkin method by adding terms proportional to the residual of Eq. (6). Here we present their formulation in a general setting that encompasses the SUPG [10], GLS [9] and Franca–Frey–Hughes (FFH) [8] formulations.

The variational form is stated in terms of these function spaces:

$$\mathscr{S} = \{u \in H^1(\Omega) | u = g \text{ on } \Gamma_g\}, \tag{12}$$

$$\mathscr{V} = \{w \in H^1(\Omega) | w = 0 \text{ on } \Gamma_g\}. \tag{13}$$

Let $\mathscr{S}^h \subset \mathscr{S}$, $\mathscr{V}^h \subset \mathscr{V}$ be finite element spaces consisting of continuous piecewise polynomials of order p. The objective is to find $u^h \in \mathscr{S}^h \ni$

$$B_{\text{SM}}(w^h, u^h) = L_{\text{SM}}(w^h)\ \forall w^h \in \mathscr{V}^h, \tag{14}$$

where

$$B_{\text{SM}}(w^h, u^h) \equiv B(w^h, u^h) + (\tau \mathbf{L} w^h, \mathscr{L} u^h), \tag{15}$$

$$L_{\text{SM}}(w^h) \equiv L(w^h) + (\tau \mathbf{L} w^h, f), \tag{16}$$

$$B(w, u) \equiv (\nabla w, -\boldsymbol{a}u + \kappa \nabla u)_\Omega + (w, a_n^+ u)_{\Gamma_h}, \tag{17}$$

$$L(w) \equiv (w, f)_\Omega + (w, h)_{\Gamma_h}, \tag{18}$$

and where
1. $\mathbf{L} = \mathscr{L}_{\mathrm{adv}}$ for SUPG,
2. $\mathbf{L} = \mathscr{L}$ for GLS,
3. $\mathbf{L} = -\mathscr{L}^*$ for FFH.

The positive parameter τ has dimensions of time and its optimal value for linear elements is given by

$$\tau = \frac{h}{2|\boldsymbol{a}|}\xi(\alpha), \tag{19}$$

$$\xi(\alpha) = \coth\alpha - \frac{1}{\alpha}. \tag{20}$$

4. Error formulation

The error is the difference between the exact solution u and the stabilized method solution u^h, i.e.,

$$e = u - u^h. \tag{21}$$

Thus $e \in \mathscr{E}$, where

$$\mathscr{E} \equiv \mathscr{S} - \mathscr{S}^h = \mathscr{V} - \mathscr{V}^h. \tag{22}$$

To derive a variational equation for the error, we begin with the variational equation for the continuous problem: Find $u \in \mathscr{S} \ni$

$$B(w,u) = L(w) \quad \forall w \in \mathscr{V}. \tag{23}$$

The stabilized methods satisfy consistency, i.e.,

$$B_{\mathrm{SM}}(w^h, e) = 0. \tag{24}$$

Substituting

$$u = u^h + e, \tag{25}$$

$$w = w^h + \tilde{w}, \quad w^h \in \mathscr{V}^h, \ \tilde{w} \in \mathscr{E} \tag{26}$$

in Eq. (23), using Eqs. (14) and (24), and dropping the tilde ($\sim$) from $\tilde{w}$ for convenience, we have: Find $e \in \mathscr{E} \ni$

$$B(w,e) = L(w) - B(w,u^h) \quad \forall w \in \mathscr{E}. \tag{27}$$

Note that both the trial solution e and the weighting function w for the error problem are chosen from the same space $\mathscr{E}$.

The term $B(w,u^h)$ in Eq. (27) can be expressed as a sum of integrations over element areas, integrated by parts on every element and rearranged to obtain, finally,

$$B(w,u^h) = (w, \mathscr{L}u^h)_{\tilde{\Omega}} - (w, [\![\hat{\boldsymbol{n}} \cdot \kappa \nabla u^h]\!])_{\Gamma_I} + (w, -a_n^- u^h + \hat{\boldsymbol{n}} \cdot \kappa \nabla u^h)_{\Gamma_h}, \tag{28}$$

where $\tilde{\Omega}$ is the sum of element interiors and Γ_I is the inter-element boundary, i.e., it is the set of all the element edges that are not on Γ. $[[\ldots]]$ denotes the jump across an inter-element edge. For more details, see [4].

Substituting Eqs. (18) and (28) in Eq. (27) and defining

$$r \equiv f - \mathscr{L}u^h, \tag{29}$$

$$r_h \equiv h + a_n^- u^h - \hat{\boldsymbol{n}} \cdot \kappa \nabla u^h, \tag{30}$$

where r is the *interior residual* and r^h is the *boundary residual*, we get

$$B(w,e) = (w,r)_{\tilde{\Omega}} + (w,[[\hat{\boldsymbol{n}} \cdot \kappa \nabla u^h]])_{\Gamma_I} + (w,r_h)_{\Gamma_h}. \tag{31}$$

The above equation is the variational equation for the global error. One can solve this equation by employing the usual finite element approach and using elements of order greater than p. However, this is expensive because it implies solving a system of equations that is at least as large as the system solved to obtain u^h. The element residual method proceeds by localizing the problem to an element level so that one only has to solve truly small systems (3×3 for linear triangles) for every element. This also makes the method amenable to parallelization.

Let $\mathscr{E}_k$ denote the restriction of $\mathscr{E}$ to Ω_k. Then the local problem can be stated as: Find $e \in \mathscr{E}_k \ni$

$$B^k(w,e) = R^k(w) \quad \forall w \in \mathscr{E}_k, \tag{32}$$

where

$$R^k(w) = (w,r)_{\Omega_k} + \alpha_{kl}\left(w,[[\hat{\boldsymbol{n}} \cdot \kappa \nabla u^h]]\right)_{\Gamma_I \cap \Gamma_k} + (w,r_h)_{\Gamma_h \cap \Gamma_k} \tag{33}$$

and in which Γ_k is the boundary of the kth element and α_{kl} is the splitting factor for the jump in the normal derivative on the edge Γ_{kl} shared by adjacent elements k and l.

The choice of α_{kl} is critical to the accuracy of the local problem. The simplest alternative is to split the jump term equally between the two elements i.e., $\alpha_{kl} = \frac{1}{2}$. A systematic scheme for flux balancing on element boundaries that substantially increases the quality of the local and global effectivity indices is presented in [1]. This algorithm has been implemented in the current work.

The continuous formulation (32) can be discretized by choosing a finite-dimensional space, $\mathscr{E}_k^h \subset \mathscr{E}_k$. Let n_b be the dimension of the space $\mathscr{E}_k^h$. The basis of $\mathscr{E}_k^h$ in 1-D consists of bubble functions as shown in Fig. 1. The minimal basis of $\mathscr{E}_k^h$ for a linear triangle is comprised of three bump functions corresponding to the three edges of a triangle. A bump function for a particular edge behaves like a quadratic bubble along that edge and decays linearly to zero perpendicular to that edge.

Thus, the discrete approximation for the error is: Find $e^h \in \mathscr{E}_k^h \ni$

$$B^k(w^h,e^h) = R^k(w^h) \quad \forall w^h \in \mathscr{E}_k^h. \tag{34}$$

Once e^h has been computed, the element error indicator Θ_k, the global error estimator Θ and the effectivity index ρ are computed in an appropriate norm in the usual fashion.

Since the local error problem (34) inherits the same bilinear operator as in Eq. (17), the corresponding matrix problem is singular in the advection-dominated limit. We add a term proportional

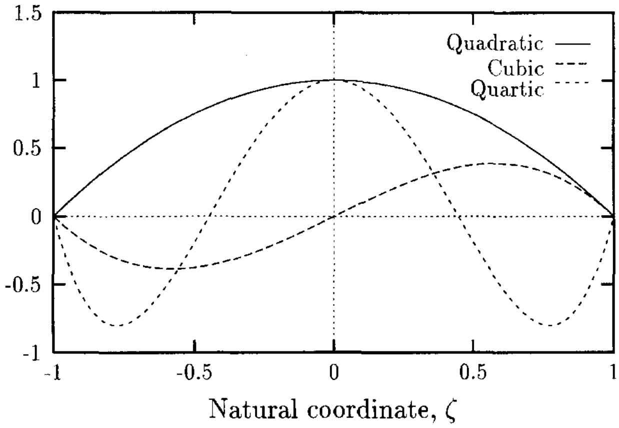

Fig. 1. Basis functions for $\mathscr{E}_k^h$ in 1-D.

to the residual of the PDE for the error ($\mathscr{L}e = r$) to impart stability to this local problem. We call this new method the stabilized element residual method (SERM). It is stated as: Find $e^h \in \mathscr{E}_k^h \ni$

$$B_{\text{SERM}}^k(w^h, e^h) = R_{\text{SERM}}^k(w^h) \quad \forall w^h \in \mathscr{E}_k^h, \tag{35}$$

$$B_{\text{SERM}}^k(w^h, e^h) \equiv B^k(w^h, e^h) + (\lambda \mathbf{L} w^h, \mathscr{L} e^h)_{\Omega_k}, \tag{36}$$

$$R_{\text{SERM}}^k(w^h) \equiv R^h(w^h) + (\lambda \mathbf{L} w^h, r)_{\Omega_k}, \tag{37}$$

where λ is a design parameter that can be selected to satisfy an appropriate design criterion. The role of λ is similar to that of τ, but its value is, in fact, quite distinct from τ as will be shown in the next section. The three choices for $\mathbf{L}$ give three separate versions of SERM. The choice of $\mathbf{L}$ here need not correspond with the choice of $\mathbf{L}$ to obtain u^h. We nondimensionalize λ as

$$\lambda = \frac{h}{2|\boldsymbol{a}|}\sigma(\alpha). \tag{38}$$

5. Selection of design parameter, σ

The one-dimensional homogeneous advection–diffusion equation

$$au_{,x} - \kappa u_{,xx} = 0 \tag{39}$$

provides a proper setting for the selection of the nondimensional design parameter, $\sigma(\alpha)$, because the exact solution is available in closed form. τ in Eq. (19) was first derived for this equation and later extended to more complicated cases. The exact solution is given by

$$u = C_1 + C_2 \exp\left(\frac{ax}{\kappa}\right), \tag{40}$$

where C_1 and C_2 are constants that can be determined by the boundary conditions. For element k, $x \in [x_k, x_{k+1}]$. Thus,

$$x = x_k + \tfrac{1}{2}(\zeta + 1)h \quad \text{where } \zeta \in [-1, 1]. \tag{41}$$

Substituting the above in Eq. (40) and introducing

$$\tilde{C}_2 = C_2 \exp\left(\frac{ax_k}{h}\right) \tag{42}$$

we can write

$$u = C_1 + \tilde{C}_2 \exp(\alpha(\zeta + 1)). \tag{43}$$

For this homogeneous problem, the stabilized methods give a nodally exact solution. Hence it is possible to write the linear finite element solution in closed form:

$$u^h = C_1 + \tilde{C}_2\left[\tfrac{1}{2}(\zeta + 1)\exp(2\alpha) - \tfrac{1}{2}(\zeta - 1)\right]. \tag{44}$$

The true error can accordingly be written as

$$e(\zeta) = \tilde{C}_2\left[\exp(\alpha(\zeta + 1)) - \tfrac{1}{2}(\zeta + 1)\exp(2\alpha) + \tfrac{1}{2}(\zeta - 1)\right]. \tag{45}$$

An analytical expression for the estimated error can be obtained as follows. Consider first the SUPG version of the local error problem obtained by replacing $\mathbf{L}$ with $\mathscr{L}_{\text{adv}}$ in Eq. (35): Find $e^h \in \mathscr{E}_k^h \ni$

$$B^k(w^h, e^h) + (\lambda\mathscr{L}_{\text{adv}}w^h, \mathscr{L}e^h)_{\Omega_k} = (w^h + \lambda\mathscr{L}_{\text{adv}}w^h, r)_{\Omega_k} \quad \forall w^h \in \mathscr{E}_k^h. \tag{46}$$

The bubble functions, depicted in Fig. 1, are zero at the ends of the element and so the boundary residual and inter-element jump terms have dropped out. Let $n_b = 1$, i.e., we use only the quadratic bubble to estimate the error. Thus

$$e^h = e_1^h(1 - \zeta^2). \tag{47}$$

It can be shown that

$$e^h(\zeta) = -\frac{\tilde{C}_2\alpha(\exp(2\alpha) - 1)}{4(1 + \alpha\sigma)}(1 - \zeta^2). \tag{48}$$

Note that the estimated error depends upon the choice of σ. Thus we can select σ by "equating" the estimated error, $e^h(\zeta)$ to the true error, $e(\zeta)$. This can be achieved in many ways:

Criterion I. The estimated error and the true error are equal at the middle of the element, i.e.,

$$e^h(\zeta = 0) = e(\zeta = 0). \tag{49}$$

Since

$$e^h(\zeta = 0) = -\frac{\tilde{C}_2\alpha(\exp(2\alpha) - 1)}{4(1 + \alpha\sigma)},$$

$$e(\zeta = 0) = -\frac{\tilde{C}_2}{2}(\exp(\alpha) - 1)^2,$$

we get

$$\sigma = \tfrac{1}{2}\coth\tfrac{1}{2}(\alpha) - 1/\alpha \equiv \sigma_{\text{mid}}^{\text{SUPG}}. \tag{50}$$

Note the similarities of the above expression with that of $\xi(\alpha)$ in (20).

Criterion II. The element effectivity index in $\|\cdot\|_{L_2}$ is equal to the ideal value of one, i.e.,

$$\rho_{L_2} = \frac{\|e^h\|_{L_2}}{\|e\|_{L_2}} = 1. \tag{51}$$

Since

$$\|e^h\|_{L_2}^2 = \tilde{C}_2^2 \alpha^2 \frac{(\exp(2\alpha) - 1)^2}{15(1 + \alpha\sigma)^2}, \qquad \|e\|_{L_2}^2 = \frac{\tilde{C}_2^2}{6\alpha^2} p(\alpha),$$

where

$$p(\alpha) = (4\alpha^2 - 9\alpha + 6)\exp(4\alpha) + (4\alpha^2 - 12)\exp(2\alpha) + (4\alpha^2 + 9\alpha + 6), \tag{52}$$

it follows that

$$\sigma = \frac{\alpha(\exp(2\alpha) - 1)}{\sqrt{2.5\,p(\alpha)}} - \frac{1}{\alpha} \equiv \sigma_{L_2}^{\mathrm{SUPG}}. \tag{53}$$

Criterion III. The element effectivity index in $|\cdot|_{H^1}$ is one, i.e.,

$$\rho_{H^1} = \frac{|e^h|_{H^1}}{|e|_{H^1}} = 1. \tag{54}$$

Since

$$|e^h|_{H^1}^2 = \tilde{C}_2^2 \alpha^2 \frac{(\exp(2\alpha) - 1)^2}{6(1 + \alpha\sigma)^2},$$

$$|e|_{H^1}^2 = \tfrac{1}{2}\tilde{C}_2^2[\alpha(\exp(2\alpha) - 1)(\exp(2\alpha) + 1) - (\exp(2\alpha) - 1)^2],$$

we obtain

$$\sigma = \frac{1}{\sqrt{3\sigma\xi}} - \frac{1}{\alpha} \equiv \sigma_{H^1}^{\mathrm{SUPG}}. \tag{55}$$

These different values of σ are plotted in Fig. 2(a) along with $\xi(\alpha)$. In all three cases, σ goes to zero linearly in the diffusion-dominated limit and asymptotes to a limiting value in the advection-dominated limit, as expected.

Recall that the previous analysis was done for the SUPG version of the SERM. We can perform a similar analysis for the GLS and FFH versions. The GLS version of the local error problem (35), obtained by replacing $\mathbf{L}$ with $\mathscr{L}$ is: Find $e^h \in \mathscr{E}_k^h \ni$

$$B^k(w^h, e^h) + (\lambda \mathscr{L} w^h, \mathscr{L} e^h)_{\Omega_k} = (w^h + \lambda \mathscr{L} w^h, r)_{\Omega_k} \quad \forall w^h \in \mathscr{E}_k^h. \tag{56}$$

Solving the local problem, again with just one bubble, we obtain

$$e^h(\zeta) = -\frac{\tilde{C}_2 \alpha(\alpha + 3\sigma)(\exp(2\alpha) - 1)}{4(\alpha + (\alpha^2 + 3)\sigma)}(1 - \zeta)^2. \tag{57}$$

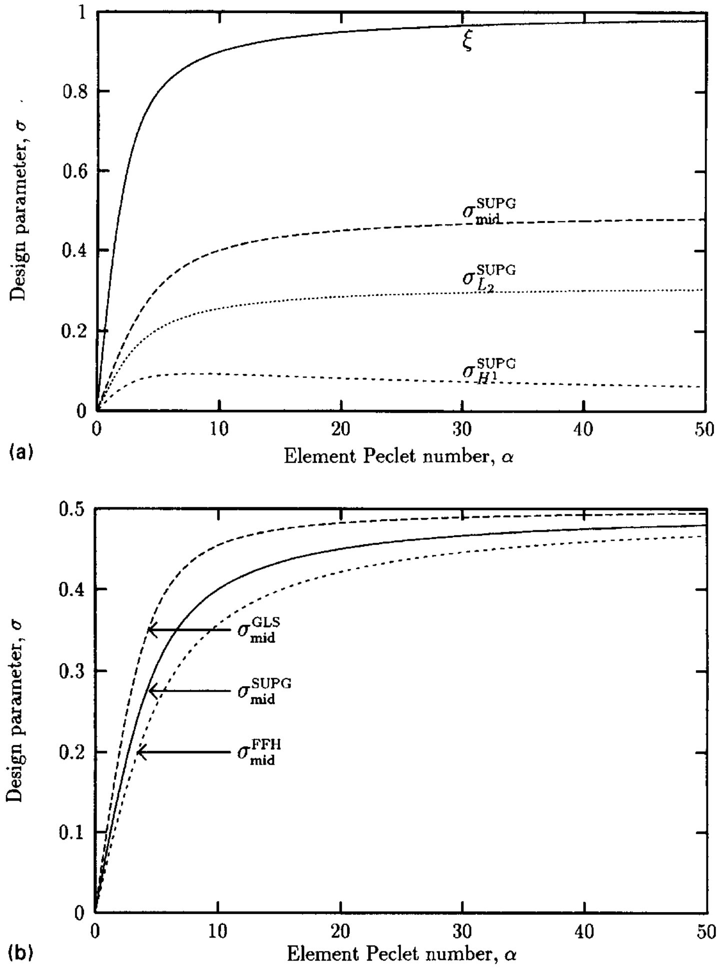

Fig. 2. A comparison of the nondimensional design parameter σ obtained from (a) different criteria for the SUPG version, and (b) criteria I for different versions of SERM.

Applying criterion I yields

$$\sigma = \frac{\frac{1}{2}\coth\frac{\alpha}{2} - \frac{1}{\alpha}}{1 - \frac{3}{\alpha}\left(\frac{1}{2}\coth\frac{\alpha}{2} - \frac{1}{\alpha}\right)} \equiv \sigma_{\mathrm{mid}}^{\mathrm{GLS}}. \tag{58}$$

It is easy to see that

$$\sigma_{\mathrm{mid}}^{\mathrm{GLS}} = \frac{\sigma_{\mathrm{mid}}^{\mathrm{SUPG}}}{1 - \frac{3}{\alpha}\sigma_{\mathrm{mid}}^{\mathrm{SUPG}}}. \tag{59}$$

In fact, this relationship is always valid irrespective of the criterion used to obtain σ. Thus

$$\sigma^{\mathrm{GLS}} = \frac{\sigma^{\mathrm{SUPG}}}{1 - \frac{3}{\alpha}\sigma^{\mathrm{SUPG}}}. \tag{60}$$

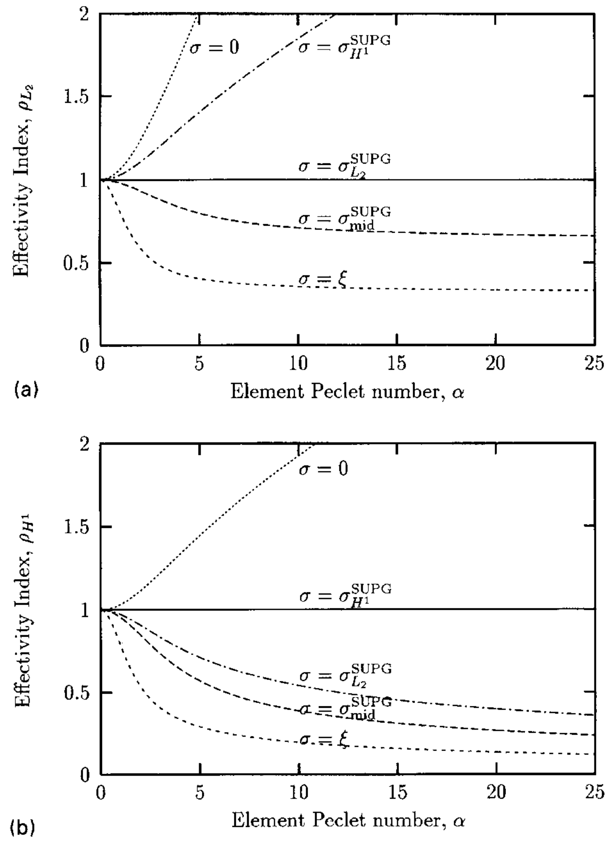

Fig. 3. Effectivity index in the (a) L_2 norm, and (b) H^1 seminorm for various values of σ.

Similarly, the design parameter for the FFH version is related to that of the SUPG version irrespective of the criterion used.

$$\sigma^{\text{FFH}} = \frac{\sigma^{\text{SUPG}}}{1 + \frac{3}{\alpha}\sigma^{\text{SUPG}}}. \tag{61}$$

A comparison of the expressions for σ, when criterion I is applied to the three versions of SERM, is shown in Fig. 2(b). We see that the three σ's are quite similar in nature. Their asymptotic value in the advection-dominated limit is identical. This is expected because both the GLS and the FFH operators reduce to the SUPG operator in the absence of diffusion.

The expressions derived for σ are identical for all elements because σ is independent of $\tilde{C}_2$. Thus, if α is uniform for all elements then the element effectivity indices are also uniform. This implies that the global effectivity index is equal to the element effectivity indices. The effectivity indices in the L_2 norm and H^1 seminorm are plotted in Fig. 3. The choice $\sigma = \xi$ corresponds to the case where σ is simply set to the value used in computing u^h. The error is under-estimated with this choice.

We also compare our results with the estimator obtained from solving the following local problem: Find $\psi^h \in \mathscr{E}_k^h \ni$

$$(\nabla w^h, \nabla \psi^h)_{\Omega_k} = R^k(w^h) \quad \forall w^h \in \mathscr{E}_k^h. \tag{62}$$

The above approach is consistent with that proposed by Ainsworth and Oden [1] for unsymmetric problems, in which they choose a local bilinear form which is coercive with respect to the original bilinear form but involves arbitrary coefficients. The case $\sigma=0$ in Fig. 3 corresponds to this technique because with just one quadratic bubble the contribution of the advection term to the matrix error problem is zero. The error is over-estimated with this choice.

6. Numerical examples

Example 1. Let us now solve the nonhomogeneous AD equation

$$au_{,x} - \kappa u_{,xx} = \sin(2\pi x) \tag{63}$$

with $x \in [0,1]$, $\kappa = 1$, and $N^{el} = 10$. The exact solution for two different values of α is plotted in Fig. 4. The exact solution has a boundary layer near $x=1$ which gets thinner as $\alpha \to \infty$ and becomes more difficult to resolve with a uniform mesh.

The global effectivity indices in the H^1 seminorm, obtained with the same choices for σ, as for the model problem in the previous section, are presented in Table 1 for various values of α. The table clearly shows that $\sigma = 0$ and $\sigma = \xi$ perform poorly as in the model problem. The choice

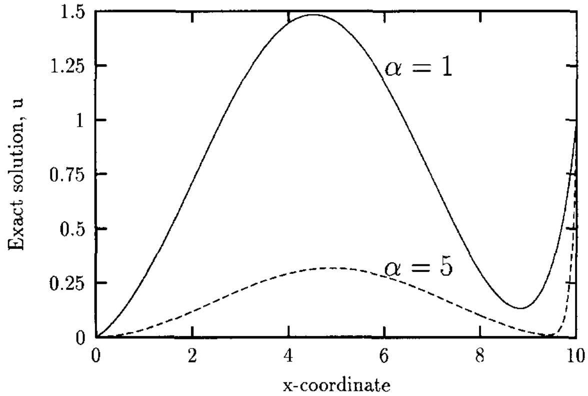

Fig. 4. The exact solution to the 1-D AD equation with a sinusoidal source for two different values of α.

Table 1
Variation of global effectivity index in the H^1 seminorm with α for different choices of σ

α	$\sigma = \sigma_{H^1}^{\mathrm{SUPG}}$	$\sigma = 0$	$\sigma = \xi$
0.5	1.000	1.003	0.978
1.0	1.004	1.029	0.838
2.5	1.004	1.165	0.479
5.0	1.003	1.443	0.296
25.0	1.049	3.089	0.124

of $\sigma = \sigma_{H^1}^{\mathrm{SUPG}}$ does not yield unit effectivity indices anymore because for nonzero f the stabilized methods do not yield a nodally exact solution. Therefore, the error is not zero on the nodes and cannot be truly represented by the bubbles alone. However, the results indicate that the optimal value derived for the homogeneous case provides surprisingly good results for the nonhomogeneous case as well.

Example 2. Let us now consider a 2-D example [16] on a square domain, $\Omega=[0,64]^2$. The parameters are

$$\kappa = 0, \qquad a = (y + 16, -x), \qquad f = 0,$$

and the boundary conditions are

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } y = 0 \quad \text{and} \quad x = 64,$$
$$u = 10 \quad \text{on } y = 64,$$
$$u = F(y) \quad \text{on } x = 0,$$

where $F(y)$ is plotted in Fig. 5. Adaptive meshes are generated using an advancing front mesh generator written by Peraire et al. [14]. The SERM error estimator is compared with the error estimator described by Eq. (62) on the sequence of adaptive meshes shown in Fig. 6. The numerical solution on the final mesh is shown in Fig. 7. A comparison of the global effectivity indices in the L_2 norm is presented in Fig. 8.

The results indicate that although the estimator in Eq. (62) provides an upper bound, it overestimates the error significantly. In contrast, the SERM estimator is closer to the ideal value of one than the previous estimator.

Here we have used $\sigma = \sigma_{L_2}^{\mathrm{SUPG}}$. Since this was derived for the 1-D bubble, this is not expected to be optimal for this 2-D problem where the basis for the error space, $\mathscr{E}_k^h$ consists of bump functions that are not zero on the entire boundary of an element. Nevertheless, we find the results encouraging and are working on computing optimal parameters for 2-D problems.

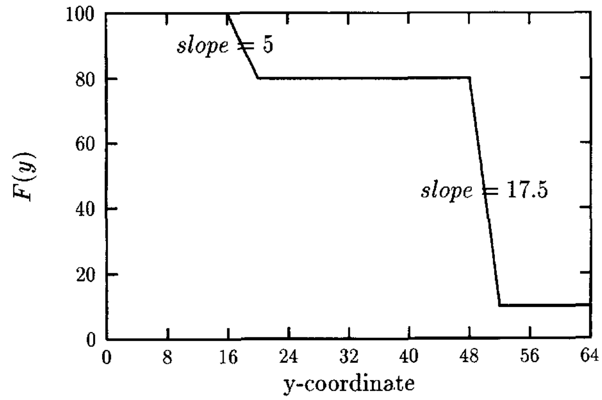

Fig. 5. Variation of the Dirichlet boundary condition along the edge $x = 0$.

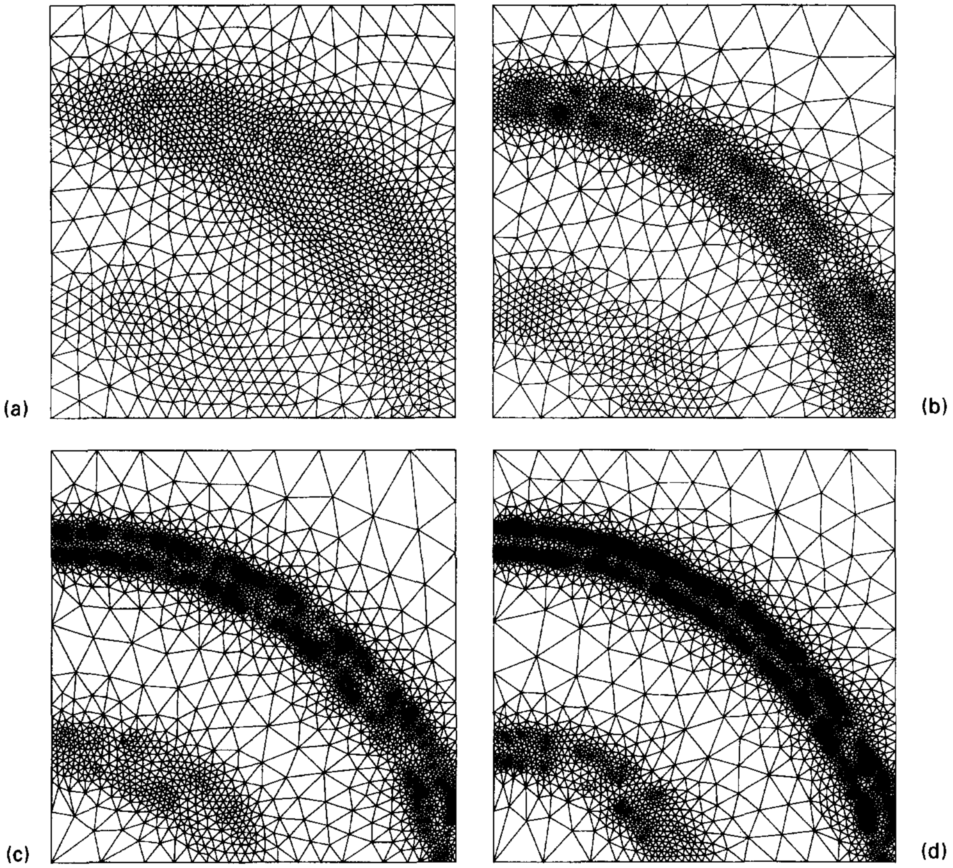

Fig. 6. A sequence of adaptive meshes for the pure advection problem with (a) 1484 nodes and 2881 elements, (b) 1897 nodes and 3701 elements, (c) 2577 nodes and 5056 elements, and (d) 3464 nodes and 6821 elements.

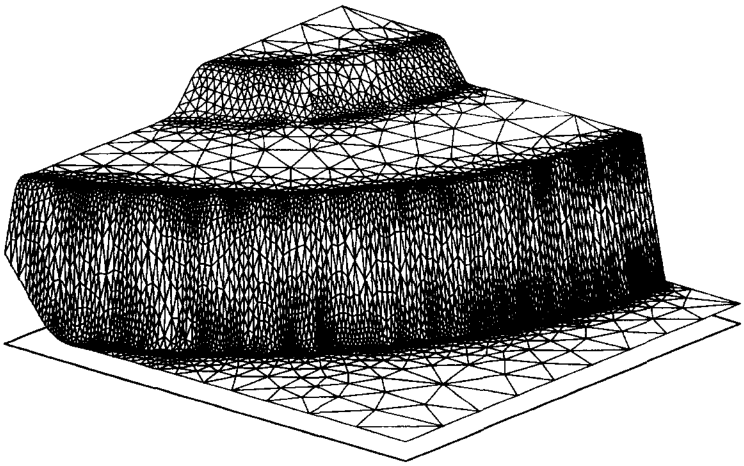

Fig. 7. The numerical solution for the pure advection problem on the final mesh (with 3464 nodes and 6821 elements).

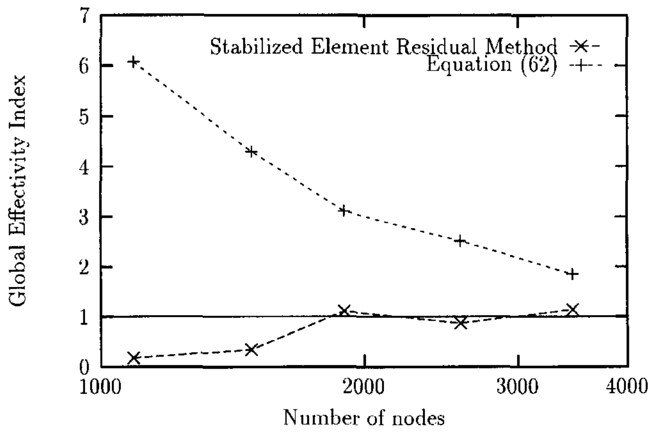

Fig. 8. Global effectivity indices in the L_2 norm for a sequence of adaptive meshes.

7. Conclusions

In this chapter we have presented an error estimation technique for the advection–diffusion equation which is based on the element residual method. The estimation technique is augmented by the inclusion of additional stabilizing terms which are proportional to the residual of the Euler–Lagrange equation corresponding to the local error problem. The approach is motivated by the success of stabilized methods such as SUPG, GLS and FFH. The stabilization parameter provides flexibility for tuning the method to obtain unit effectivity indices under certain conditions. The optimal value of this parameter depends upon the choice of the operator for the weighting function and the choice of the norm used to measure the error.

For two-dimensional problems, the error space consists of bump functions and in this case the stabilization parameters obtained from the one-dimensional analysis will not be optimal. Nevertheless, the application of these parameters to two-dimensional problems has produced good results. We are currently working on the multidimensional generalization of the proposed method. Although the work presented here is valid only for the AD equation, it has been presented in a framework that is extensible to other equations.

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